PUTNAM PRACTICE SET 1

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Problem 1. Let m be a positive integer, let a be a positive real number and let θ be a real number. Prove there exist m quadratic polynomials $Q_1(x), \dots, Q_m(x) \in \mathbb{R}[x]$ such that

$$x^{2m} - 2a^m x^m \cos(m\theta) + a^{2m} = Q_1(x) \cdot Q_2(x) \cdots Q_m(x).$$

Problem 2. For a matrix, the following operations are considered acceptable:

- change the sign of each entry on any given row;
- change the sign of each entry on any given column;
- switch two rows;
- switch two columns.

Prove that we cannot use the above operations to transform the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 \end{pmatrix}$$

into the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 \end{pmatrix}.$$

Problem 3. Find the infimum of $a^2 + b^2$ over all the possible pairs (a, b) of real numbers with the property that the equation

$$x^4 + ax^3 + bx^2 + ax + 1 = 0$$

has 4 distinct real roots.

Problem 4. Let k be a positive integer. Find the set of all tuples $(a_1, \dots, a_{k+1})$ of non-negative integers satisfying the following properties:

- $a_1 = 0$.
- $|a_i - a_{i+1}| = 1$ for $i = 1, \dots, k$.

Problem 5. Prove that for each positive integer n, we have

$$2^n \cdot \prod_{k=1}^{n} \sin\left(\frac{k\pi}{2n+1}\right) = \sqrt{2n+1}.$$

Problem 6. Let P be a set of 5 distinct prime numbers and let B be the set of all 15 numbers obtained as a product of two numbers from P, not necessarily distinct. We partition B into 5 disjoint sets $C_1, \cdots, C_5$, each one of them containing precisely 3 elements from B, and moreover having the property that for each $i = 1, \ldots, 5$, there is a prime dividing each of the 3 numbers from the set C_i. How many possible partitions of B into 5 such subsets are there?